# A NOTE ON STAMPING

ANNEKE BART AND KEVIN P. SCANNELL

Abstract. The stamping deformation was defined by Apanasov as the first example of a deformation of the flat conformal structure on a hyperbolic 3-orbifold distinct from bending. We show that in fact the stamping cocycle is equal to the sum of three bending cocycles. We also obtain a more general result, showing that geodesic lengths are constant to first order under deformations of the flat conformal structure for any hyperbolic 3-orbifold.

## 1. Introduction

Let $\Gamma$ be a lattice in $\mathrm{Isom}\mathbb{H}^3$ and let $M = \Gamma\backslash\mathbb{H}^3$ be the corresponding hyperbolic 3-orbifold. We are interested in the so-called "quasi-Fuchsian" deformation theory of $\Gamma$ when included into the larger group $\mathrm{Isom}\mathbb{H}^4$. By the holonomy theorem and the isomorphism $\mathrm{Isom}\mathbb{H}^4 \cong \mathrm{Möb}\mathbb{S}^3$, local deformations of $\Gamma$ are equivalent to deformations of the induced flat conformal (or Möbius) structure on $M$.

The bending construction is the primary source of deformations of this kind, but works only in the case that $M$ contains an embedded two-sided totally geodesic surface $S$. Under small bending deformations, one component of the convex hull boundary in $\mathbb{H}^4$ of the resulting quasi-Fuchsian group is 3-dimensional and totally geodesic except along codimension-one singularities corresponding to copies of $\mathbb{H}^2 \cong \tilde{S}$ where it is "bent". Examples of this kind first appeared in work of Thurston [14, §8.7.3] and Apanasov-Tetenov [4], [1]; a general existence result holding in all dimensions was proved by Johnson-Millson [9].

There are only a few known deformations that produce codimension-two singularities in the convex hull boundary. The first of these, and the example of primary interest in this paper, was the so-called "stamping deformation" defined by Apanasov [2], [3] for a certain finite-covolume lattice in $\mathrm{Isom}\mathbb{H}^3$. A few years later M. Kapovich [10] computed the dimension of the deformation space for any cocompact reflection group (it's the number of faces minus 4) and showed that the base representation into $\mathrm{Isom}\mathbb{H}^3$ is a smooth point of the representation variety. He also gave an example of a non-tetrahedral reflection group containing no embedded totally geodesic surfaces (see [10, §6.3]); it follows that the deformations of this group have codimension-two singularities in the convex hull boundary.

Very little else is known about this question in general, though there are a few other interesting examples in the literature; some rigid [12], and some admitting deformations [13], [5], [11], [6] (occasionally only to first order).

Our interest in the stamping example began with [6], in which we found an infinitesimal deformation of the link complement $8^2_{14}$ supported on a piecewise totally geodesic 2-complex that is not isotopic to an immersed totally geodesic surface. This complex contains a "singular geodesic" formed by the intersection of three 2-cells along their boundaries, arranged combinatorially like three pages meeting the binding of a book. In this, our example is somewhat evocative of Apanasov's stamping example which has a similar codimension-two singularity, though in his case it is formed as the intersection of three *complete* two-dimensional planes passing through the singularity.

*Date*: October 1, 2005.

1991 *Mathematics Subject Classification.* Primary: 57M50; Secondary: 57N10, 22E40.

*Key words and phrases.* stamping, bending, flat conformal structures, quasi-Fuchsian groups.

This difference turned out to be of critical importance in our failed attempts to use stamping as a recipe for integrating our infinitesimal deformation. Indeed, we were led directly to the following result, which we believe helps clarify the general picture:

**Theorem 1.1.** *The stamping deformation is a sum of bending deformations.*

We give a precise formulation and proof in §3.

We were led to Theorem 1.1 by the following simple observation that may be of independent interest (see Proposition 2.1): lengths of geodesics in $M$ are constant to first order under infinitesimal deformations of the flat conformal structure. This is basically a consequence of the local rigidity of lattices in $\mathrm{O}(3,1)$.

## 2. Derivatives of lengths of geodesics

For $g \in \mathrm{Isom}\mathbb{H}^n$, we write

$$\ell(g) = \inf_{x \in \mathbb{H}^n} \{d_{\mathbb{H}^n}(x, g \cdot x)\}$$

for the minimum translation length of $g$. Of course this is a conjugacy invariant:

$$\ell(ghg^{-1}) = \ell(h)$$

for all $g, h \in \mathrm{Isom}\mathbb{H}^n$.

In all that follows we will be considering the case of a lattice $\Gamma \subseteq G_3 = \mathrm{Isom}\mathbb{H}^3$ and its deformations in the larger group $G_4 = \mathrm{Isom}\mathbb{H}^4$. Of course $\mathrm{Isom}\mathbb{H}^n$ is locally isomorphic to $SO(n,1)$, so we have identifications of Lie algebras $\mathfrak{g}_3 \cong \mathfrak{so}(3,1)$ and $\mathfrak{g}_4 \cong \mathfrak{so}(4,1)$.

Now if $\rho_t$ is family of representations in $\mathrm{Hom}(\Gamma, G_4)$ depending smoothly on the parameter $t$, with $\rho_0$ equal to the inclusion of $\Gamma$ in $G_3$, we can view a tangent vector to the family as a function $v : \Gamma \to \mathfrak{so}(4,1)$ given by

$$v(\gamma) = \dot{\rho}(\gamma)\gamma^{-1}$$

and satisfying the cocycle relation

$$v(\gamma_1\gamma_2) = v(\gamma_1) + Ad(\gamma_1)v(\gamma_2).$$

Trivial deformations (those induced by conjugation in $G_4$) yield coboundaries $v(\gamma) = v_0 - Ad(\gamma)v_0$, and it is therefore reasonable to view $H^1(\Gamma, \mathfrak{so}(4,1))$ as the space of *infinitesimal quasi-Fuchsian deformations* of $\Gamma$.

For any $\gamma \in \Gamma$, the inclusion of the subgroup $\langle\gamma\rangle$ induces a restriction map on cohomology which we write

$$i_\gamma^* : H^1(\Gamma, \mathfrak{so}(4,1)) \to H^1(\langle\gamma\rangle, \mathfrak{so}(4,1)).$$

When $\Gamma$ has parabolic elements, it is typical to restrict one's attention to deformations that are trivial on the boundary cusps; more precisely, in such cases we shall consider the subspace $PH^1(\Gamma, \mathfrak{so}(4,1))$ consisting of cohomology classes $v$ such that $i_\gamma^*(v) = 0$ for every parabolic $\gamma \in \Gamma$.

**Proposition 2.1.** *Let $\Gamma$ be a (non-uniform) lattice in $G_3$. If $v \in (P)H^1(\Gamma, \mathfrak{so}(4,1))$ is an infinitesimal deformation of $\Gamma$ into $G_4$, then $i_\gamma^*(v) = 0$ for $\gamma \in \Gamma$ unless $\gamma$ is purely hyperbolic. In any case, we have*

$$\left.\frac{d}{dt}\right|_{t=0} \ell(\exp(tv(\gamma))\gamma) = 0.$$

*Proof.* First, the Lie algebra $\mathfrak{so}(4,1)$ splits as a $G_3$-module:

$$\mathfrak{so}(4,1) \cong \mathfrak{so}(3,1) \oplus \mathbb{R}^4_1,$$

where $\mathbb{R}^4_1$ is the standard representation of $G_3 \subseteq \mathrm{O}(3,1)$ on Minkowski space. From this we obtain a splitting in cohomology

$$(P)H^1(\Gamma, \mathfrak{so}(4,1)) \cong (P)H^1(\Gamma, \mathfrak{so}(3,1)) \oplus (P)H^1(\Gamma, \mathbb{R}^4_1).$$

Therefore

$$(P)H^1(\Gamma, \mathfrak{so}(4,1)) \cong (P)H^1(\Gamma, \mathbb{R}^4_1)$$

by local rigidity of lattices in $G_3$ [7], [8]. So in what follows we will assume that the values of $v$ lie in the standard representation $\mathbb{R}^4_1$.

If $\gamma$ is elliptic, it has finite order since $\Gamma$ is discrete, and it follows that $H^1(\langle\gamma\rangle, \mathbb{R}^4_1) = 0$. If $\gamma$ is parabolic, then $PH^1(\langle\gamma\rangle, \mathbb{R}^4_1) = 0$ by hypothesis. Finally suppose $\gamma$ is loxodromic with a non-trivial rotational part. Since $\langle\gamma\rangle$ is infinite cyclic, a cocycle is uniquely determined by its value on $\gamma$, so $Z^1(\langle\gamma\rangle, \mathbb{R}^4_1) \cong \mathbb{R}^4_1$. But $\gamma$ (viewed as an element of $\mathrm{O}(3,1)$) has no 1-eigenvalues and so $I - \gamma$ maps onto $\mathbb{R}^4_1$. This says $Z^1 = B^1$ and so $H^1(\langle\gamma\rangle, \mathbb{R}^4_1) = 0$.

For the second part, we first observe that the result is clear when $v(\gamma) = v_0 - \gamma v_0$ is a coboundary; in this case,

$$\begin{aligned} \ell(\exp(tv_0 - t\gamma v_0 + \mathrm{O}(t^2))\gamma) &= \ell(\exp(tv_0)\exp(-t\gamma v_0)\gamma) \\ &= \ell(\exp(tv_0)\gamma\exp(-tv_0)\gamma^{-1}\gamma) \\ &= \ell(\exp(tv_0)\gamma\exp(tv_0)^{-1}) \\ &= \ell(\gamma) \end{aligned}$$

and the result follows by differentiating. Therefore we need only consider the case that $\gamma$ is purely hyperbolic, and by conjugating we may assume $\gamma$ is a Möbius transformation of the form $\mathbf{x} \mapsto \lambda\mathbf{x}$ for $1 \neq \lambda \in (0, \infty)$. Since $\langle\gamma\rangle \cong \mathbb{Z}$,

$$H^1(\langle\gamma\rangle, \mathfrak{so}(4,1)) \cong H^0(\langle\gamma\rangle, \mathfrak{so}(4,1)) \cong \mathfrak{so}(4,1)^{\langle\gamma\rangle}$$

which is four-dimensional, spanned by Lie algebra elements $v_0$, $v_1$, $v_2$, $v_3$, where $\langle v_1, v_2, v_3\rangle$ form a basis of $\mathfrak{so}(3)$, $v_1 \in \mathfrak{so}(2)$, and $\exp(tv_0)$ is of the form $\mathbf{x} \mapsto e^{t\lambda_0}\mathbf{x}$, $\lambda_0 \in \mathbb{R}$. It is clear that the deformation $\exp(tv_j)\gamma$ remains in $G_3$ if and only if $j = 0, 1$. So by choosing our coefficients to lie in $\mathbb{R}^4_1$ as above, we may assume $j = 2$ or $j = 3$, and in either case $\ell(\exp(tv(\gamma))\gamma)$ is constant in $t$. □

There is a mildly subtle point in the previous argument that contrasts with the well-known fact from hyperbolic geometry that any loxodromic in $G_4$ is conjugate into $G_3$. The point is that one can have a small deformation of a purely hyperbolic element of $G_3$ through loxodromics in $G_4$, but for which there are no *small* elements conjugating back into $G_3$. This is precisely the source of the two-dimensional cohomology group $H^1(\langle\gamma\rangle, \mathbb{R}^4_1)$ for purely hyperbolic $\gamma$.

**Corollary 2.2.** *If $\rho_t : (-\epsilon, \epsilon) \to \mathrm{Hom}(\Gamma, G_4)$ is a deformation of $\rho_0 = \mathrm{id}$ depending smoothly on the parameter $t$, then $\frac{d}{dt}\big|_{t=0}\ell(\rho_t(\gamma)) = 0$ for all $\gamma \in \Gamma$.*

*Proof.* If we let $v(\gamma) = \dot\rho(\gamma)\gamma^{-1}$ be the corresponding infinitesimal deformation, then

$$\rho_t(\gamma) = \exp(tv(\gamma) + \mathrm{O}(t^2))\gamma$$

and the result follows from the previous proposition. □

## 3. Stamping is bending

We begin by reviewing the details from [3]. The initial group $\Gamma \subseteq G_3$ is generated by side pairings on a finite-volume hyperbolic polyhedron defined by a configuration of eight circles in $\partial\mathbb{H}^3 = \mathbb{S}^2 = \mathbb{R}^2 \cup \{\infty\}$. Six of these circles have radius one and are centered at the sixth roots of unity $\sqrt{3}e^{\frac{k\pi i}{3}}$; the faces defined by these circles are paired with themselves, giving generating reflections $\sigma_i$, $i = 0, \ldots, 5$. The other two faces are defined by the circles centered at the origin of radius 1 and 2; these are paired with each other, giving a purely hyperbolic generator which we denote by $\gamma$. Observe that all pairs of intersecting circles meet in angles of $\frac{\pi}{3}$; it follows easily that $\Gamma$ is a finite-covolume (non-cocompact) lattice in $G_3$. See figure 1.

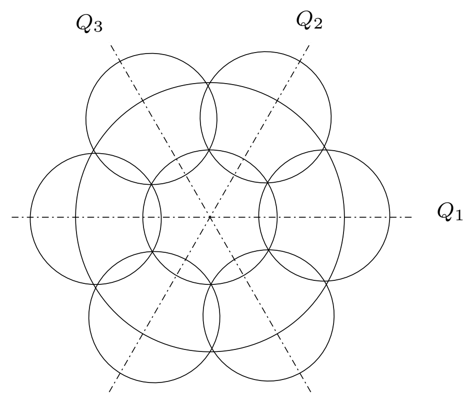


FIGURE 1. Initial configuration for the stamping example, including the bending lines $Q_j$

To define a deformation of $\Gamma$ into $G_4$, we view these circles as also defining eight 2-spheres in $\partial\mathbb{H}^4 = \mathbb{S}^3 = \mathbb{R}^3 \cup \{\infty\}$, and vary these spheres while preserving the angles between them and the structure of the cusps. For values of $t$ in the interval $(\frac{\pi}{2}, \frac{2\pi}{3}]$, construct unit vectors $v_1(t)$, $v_2(t)$, and $v_3(t)$ in $\mathbb{R}^3$ such $v_i(t) \cdot v_j(t) = \cos t$ for $i \neq j$. These vectors define a trihedral angle which is an octant for $t = \frac{\pi}{2}$ and gradually "flattens out" and becomes planar at $t = \frac{2\pi}{3}$. Clearly we have some freedom in constructing these vectors, and following Apanasov we choose to normalize so that $v_1(t) = (1, 0, 0)$ for all $t$ (there is one remaining degree of freedom coming from rotation about $v_1(t)$ that we leave unspecified for the moment).

Next, for any permutation $(ijk)$ of the indices $(123)$, we define $w_k(t)$ to be the unit vector in the direction $v_i(t) + v_j(t)$. Let $\Gamma_t$ be the group generated by reflections in spheres centered at $\sqrt{3}v_i(t)$ and $\sqrt{3}w_i(t)$ for $i = 1, 2, 3$, each of radius $r_t = 2\sin\frac{t}{4}$, together with a purely hyperbolic element pairing the spheres centered at the origin of radii $R_t^{\pm} = \sqrt{3}\cos\frac{t}{4} \pm \sin\frac{t}{4}$. Clearly $\Gamma_{\frac{2\pi}{3}} = \Gamma$. A somewhat tedious calculation verifies that $\Gamma_t$ is a discrete group isomorphic to $\Gamma$ for all $t \in (\frac{\pi}{2}, \frac{2\pi}{3}]$; we write $\rho_t : \Gamma \cong \Gamma_t \subseteq G_4$ for the composition of this isomorphism with the inclusion into $G_4$. The curve of representations $\rho_t$ is what Apanasov calls the *stamping deformation*.

Let $\mathcal{S}$ denote the space of all round 2-spheres in $\mathbb{R}^3$; we identify $\mathcal{S}$ with a half-space in $\mathbb{R}^4$ by assigning coordinates $(x, y, z, r)$ to a sphere centered at $(x, y, z)$ of radius $r$. There is a map $\mathcal{S} \to G_4$ that sends a given sphere to the Möbius reflection fixing it. The primary fact we shall need about this map is that it is smooth, and in particular that a tangent vector $(\dot{x}, \dot{y}, \dot{z}, \dot{r})$ to $\mathcal{S}$ gives rise to a well-defined (and easily computable) element of $\mathfrak{so}(4, 1)$. Tangent vectors such that $\dot{x} = \dot{y} = \dot{r} = 0$ correspond to Lie algebra elements that lie in the complement $\mathbb{R}_1^4$ of $\mathfrak{so}(3, 1)$ as discussed in the proof of Proposition 2.1.

Observe that there are three totally geodesic suborbifolds of $\Gamma\backslash\mathbb{H}^3$ defined by the planes $Q_j$ in the upper half space containing the positive $z$-axis and the point $v_j(\frac{2\pi}{3})$ for $j = 1, 2, 3$. The geodesic formed by the intersection of the planes $Q_j$ is also the invariant axis for the purely hyperbolic element $\gamma \in \Gamma$.

In [3], it is argued that a cocycle for the stamping deformation cannot lie in the linear span of the bending cocycles defined by the individual surfaces $Q_j$ because (a calculation shows) $\frac{d}{dt}\big|_{t=0}\ell(\rho_t(\gamma)) \neq 0$ which is clearly not true for a linear combination of bends. We saw in §2, however, that this derivative cannot be non-zero for *any* deformation (or any element) of $\Gamma$.

As it turns out, the problem lies with the fact that the given parameterization of stamping (in $t$) does not define a *smooth* family of representations into $G_4$, and therefore does not define a cocycle at all. However, if we reparameterize appropriately, the same curve of representations can be made smooth, thereby defining a cocycle $v \in PH^1(\Gamma, \mathfrak{so}(4, 1))$ such that, as expected, $\frac{d}{ds}|_{s=0}\ell(\rho_s(\gamma)) = 0$ with respect to the new parameter $s$. In fact, more is true:

**Theorem 3.1.** *The stamping cocycle $v$ is the sum of bending cocycles supported on the totally geodesic surfaces $Q_j$ for $j = 1, 2, 3$.*

*Proof.* We begin by showing that the given parameterization in $t$ is not smooth. If it were, one could differentiate the conditions $v_i(t) \cdot v_i(t) = 1$ and $v_i(t) \cdot v_j(t) = \cos t$ at $t = \frac{2\pi}{3}$ to obtain the following equalities, where we have used the normalization $v_1(t) = (1, 0, 0)$ and have written $v_i(t) = (x_i, y_i, z_i)$ for $i = 2, 3$:

$$
\begin{aligned}
&-\frac{1}{2}\dot{x_2} + \frac{\sqrt{3}}{2}\dot{y_2} = 0 \\
&-\frac{1}{2}\dot{x_3} - \frac{\sqrt{3}}{2}\dot{y_3} = 0 \\
&\dot{x_2} = -\frac{\sqrt{3}}{2} \\
&\dot{x_3} = -\frac{\sqrt{3}}{2} \\
&-\frac{1}{2}\dot{x_3} - \frac{1}{2}\dot{x_2} + \frac{\sqrt{3}}{2}\dot{y_3} - \frac{\sqrt{3}}{2}\dot{y_2} = -\frac{\sqrt{3}}{2}
\end{aligned}
$$

The first four equations imply that $\dot{y_3} = -\dot{y_2} = \frac{1}{2}$, but plugging these values into the fifth equation leads to $\sqrt{3} = -\frac{\sqrt{3}}{2}$, a contradiction.

With an eye toward reparameterizing, we can make this even more explicit. By symmetry we may choose

$$v_2(t) = (\cos t, y(t), z(t))$$

and

$$v_3(t) = (\cos t, -y(t), z(t))$$

from which it follows that

$$\cos^2 t + y^2 + z^2 = 1$$

and

$$\cos^2 t - y^2 + z^2 = \cos t.$$

Solving for $y$ and $z$, we obtain:

$$
\begin{aligned}
y(t)^2 &= \frac{1}{2}(1 - \cos t) \\
z(t)^2 &= \frac{1}{2} + \frac{1}{2}\cos t - \cos^2 t.
\end{aligned}
$$

Expanding the expression for $z(t)^2$ as a Taylor series, we find:

$$z(t)^2 = C(\frac{2\pi}{3} - t) + \mathrm{O}((\frac{2\pi}{3} - t)^2)$$

for $C \neq 0$ and so $z(t)$ does not have a finite derivative at $t = \frac{2\pi}{3}$.

To fix this, we simply select the new parameter $s = \sqrt{C(\frac{2\pi}{3} - t)}$, so that $t = \frac{2\pi}{3} - \frac{1}{C}s^2$. Since each of the functions $x(t) = \cos t$, $y(t)$, $z(t)$, $r_t$, and $R_t^{\pm}$ is differentiable with respect to $t$ at $t = \frac{2\pi}{3}$, the chain rule implies that each derivative with respect to $s$ exists and is equal to 0 (since $\frac{dt}{ds} = 0$). On the other hand, by construction, we have $\frac{dz}{ds}(0) = 1$, and so each of the six reflecting spheres moves in a tangent direction $(\dot{x}, \dot{y}, \dot{z}, \dot{r})$ of the form $(0, 0, \beta_k, 0)$. Note that this now agrees with the basis for $H^1$ of a reflection group

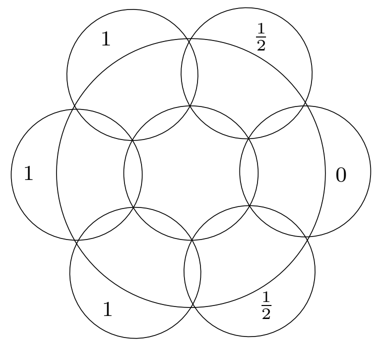


FIGURE 2. Values of $\dot{z}$ for the reparameterized stamping cocycle

exhibited by Kapovich in [10, §3.2], and corresponds to a choice of cocycle representative with coefficients in $\mathbb{R}^4_1$ as in Proposition 2.1. The values of $\beta_k$ for our chosen normalization are given in figure 2.

The bending deformations are easy to describe to first order. First, the plane $Q_1$ is the $xz$-plane, which meets two of the generating reflections in right angles, and separates the other four into two intersecting pairs. To bend along this plane we rotate each of these pairs about the $x$-axis through the same angle and "upwards" (in the positive $z$ direction) and leave the other generators alone. To first order, we may choose the speed of rotation in such a way that the rotated spheres vary by $(\dot{x}, \dot{y}, \dot{z}, \dot{r}) = (0, 0, \frac{1}{2}, 0)$, since their centers are all the same distance from the $x$-axis.

The planes $Q_2$ and $Q_3$ divide up the generating spheres similarly, in each case into two pairs of two intersecting spheres, with one pair involving the sphere centered at $v_1(t) = (1, 0, 0)$. Now in these two cases, instead of bending each side by the same amount, we choose to bend in such a way that the side containing $(1, 0, 0)$ is left unchanged while the other side is rotated upwards. Doing so contributes $(\dot{x}, \dot{y}, \dot{z}, \dot{r}) = (0, 0, \frac{1}{2}, 0)$ to each rotated sphere; once each for the spheres centered at $\sqrt{3}e^{\frac{2\pi i}{3}}$ and $\sqrt{3}e^{\frac{4\pi i}{3}}$ and twice for the sphere centered at $(-1, 0, 0)$. Summing the contributions from the bends along $Q_1$, $Q_2$, and $Q_3$ we recover exactly the same vectors $(0, 0, \beta_k, 0)$ that were computed above for the reparameterized stamping deformation (and depicted in figure 2).

For both deformations, the two 2-spheres centered at the origin are fixed to first order, and so each cocycle in $PH^1(\Gamma, \mathfrak{so}(4, 1))$ is completely determined by the corresponding 6-tuple of tangent vectors $(\dot{x}, \dot{y}, \dot{z}, \dot{r})$. Since these vectors coincide, the cocycles must coincide. □

The general phenomenon described in this theorem is not new; Kapovich gave yet another reflection group in [10, §6.2] that contains a pair of intersecting totally geodesic surfaces, such that the linear span of the bending cocycles is tangent to a smooth 2-dimensional family of deformations. However, these deformations are only given implicitly (in the literal sense that they are shown to exist by an application of the implicit function theorem). For this reason we believe the stamping example retains some interest; now instead of it being the prototypical non-bend, we hope it can be used as a blueprint for integrating sums of bending cocycles along intersecting surfaces more generally, in the spirit of Tan's paper [13]. This should be contrasted with the higher-dimensional case, where Johnson and Millson [9] showed that there are quadratic obstructions to integrating a sum of bends in some cases.

Finally, we have not been able to show that the "stamping with torsion" example from [3, §4] reduces to any simpler deformation, and probably deserves closer study.

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Department of Mathematics and Computer Science, Saint Louis University, St. Louis, MO 63103
*E-mail address*: barta@slu.edu

Department of Mathematics and Computer Science, Saint Louis University, St. Louis, MO 63103
*E-mail address*: scannell@slu.edu