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3-manifolds which are spacelike slices of flat spacetimes

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Abstract

We continue work initiated in a 1990 preprint of Mess giving a geometric parametrization of the moduli space of classical solutions to Einstein's equations in (2 + 1) dimensions with cosmological constant $\Lambda = 0$ or -1 (the case of $\Lambda = +1$ has been worked out in the interim by the present author). In this paper we make a first step toward the (3 + 1)-dimensional case by determining exactly which closed 3-manifolds M^3 arise as spacelike slices of flat spacetimes, and by finding all possible holonomy homomorphisms $\pi_1(M^3) \to ISO(3, 1)$.

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1. Introduction

This paper answers a fundamental question in classical gravity by determining all possible topologies of closed universes assuming a flat spacetime metric. This work continues two closely related lines of research; namely, the description of the moduli space of classical solutions to Einstein's equations in (2 + 1) dimensions due to Mess [20], and the classification of de Sitter spacetimes in all dimensions from [26]. The original physical motivation for these papers was provided by Witten's Chern–Simons formulation of (2 + 1)-dimensional gravity in the late 1980s [32–34]; his approach and successive attempts at quantization in the (2 + 1)-dimensional case rely on an understanding of the moduli space of classical solutions for a fixed spacetime topology $M \times \mathbb{R}$. Therefore, to even get started, one needs to know which topological types for M are possible. A good introduction to these ideas is given in Carlip's book [3], particularly chapters 2 and 4 which, among other things, reprise Mess' work (also useful are the appendices to [3] which serve as a reference for much of the mathematical terminology used in this paper).

Our expectation is that most of the results in the flat (2 + 1)-dimensional case will carry over to the higher-dimensional cases (this was also conjectured by Mess). For instance, if M is a closed spacelike slice of a flat spacetime (see the definition below), one would like to show that: (a) the linear holonomy of M is discrete, (b) there can be no topology change, (c) the spacetime metric is determined by the holonomy and (d) there is a singularity in the past or future (but not both) when M is not finitely covered by a torus. In addition to resolving the basic question of which closed 3-manifolds M can arise in this set-up, this paper makes a start at some of the above questions through a detailed analysis of possible holonomy representations.

Throughout this paper, M will denote a closed, connected 3-manifold. Our current understanding of 3-manifold topology owes a great deal to Thurston's introduction of geometric techniques in the 1970s. In particular, his geometrization conjecture [27, 29] says that given a closed 3-manifold, there is a canonical process by which it can be cut open so that the resulting pieces are *geometric*: this means they can be given Riemannian metrics which are locally isometric to one of eight simply connected Riemannian homogeneous spaces. As some of these 'model spaces' will arise in the statement of the main theorem and in the course of its proof, it is worth setting up some notation for them. There are, of course, the three constant-curvature model spaces: Euclidean space $\mathbb{E}^3$, hyperbolic space $\mathbb{H}^3$ and the 3-sphere $\mathbb{S}^3$; two product spaces $\mathbb{H}^2 \times \mathbb{R}$ and $\mathbb{S}^2 \times \mathbb{R}$; and three three-dimensional Lie groups *Nil*, *Solv* and $\widetilde{SL_2\mathbb{R}}$ equipped with natural left-invariant metrics. If X is one of these eight spaces, we say that a 3-manifold is *modelled on* X if it admits a metric locally isometric to X. When a manifold M is modelled on $\mathbb{H}^3$, we often say simply that M is *hyperbolic*. If we exclude $\mathbb{H}^3$ and *Solv*, then M is modelled on one of the six remaining spaces if and only if M is a *Seifert fibre space* [27, theorem 5.3].

We say that M is a *spacelike slice of a flat spacetime* if there is a $(3 + 1)$-dimensional Lorentzian manifold N locally isometric to Minkowski space $\mathbb{R}^4_1$ and an embedding $f : M \hookrightarrow N$ such that $f(M)$ is spacelike and has trivial normal bundle. This notion is a bit more general than that of a partial Cauchy hypersurface, since we do not exclude the possibility that the ambient spacetime N has timelike curves meeting $f(M)$ multiple times. One immediate consequence of our main theorem is that a spacelike slice of a flat spacetime is geometric, and, in fact, can only be modelled on three of the eight model spaces:

Theorem 1.1. *M is a spacelike slice of a flat spacetime if and only if M is modelled on $\mathbb{H}^3$, $\mathbb{E}^3$ or $\mathbb{H}^2 \times \mathbb{R}$.*

The manifolds modelled on $\mathbb{E}^3$ and $\mathbb{H}^2 \times \mathbb{R}$ are understood completely. There are exactly ten closed 3-manifolds modelled on $\mathbb{E}^3$ [35, section 3.5], all of which are finitely covered by the 3-torus $T^3 = \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$. Though there are infinitely many 3-manifolds modelled on $\mathbb{H}^2 \times \mathbb{R}$, each is finitely covered by $\Sigma \times \mathbb{S}^1$, where Σ is a closed surface of negative Euler characteristic (see lemma 2.5 below). Thus we have as a corollary:

Corollary 1.2. *If M is a spacelike slice of a flat spacetime and is not hyperbolic, then a finite cover of M is homeomorphic to $\Sigma \times \mathbb{S}^1$, where Σ is a closed, orientable surface, $\Sigma \not\cong \mathbb{S}^2$.*

The class of hyperbolic 3-manifolds is much richer than the others by the work of Thurston, which shows that 'most' 3-manifolds are hyperbolic. In fact, the geometrization conjecture predicts that any closed, irreducible 3-manifold with infinite fundamental group not containing a $\mathbb{Z} \oplus \mathbb{Z}$ subgroup is hyperbolic.

Mess observes [20, p 55] that the $(2+1)$-dimensional case generalizes in part to the $(3+1)$-dimensional case; in particular, he claims the following result which will follow immediately from our proof of theorem 1.1:

Corollary 1.3. *Let M be a spacelike slice of a flat spacetime. If the linear holonomy representation $L : \pi_1(M) \to O(3, 1)$ is irreducible, then M is hyperbolic.*

Also noteworthy is the work of Waelbroeck [30] which is closely related to this paper and [20]. Flat spacetimes homeomorphic to $M \times \mathbb{R}$ are studied in terms of solutions of the so-called $B \wedge F$ theory. Our main result shows, however, that many of the solutions found in [30] (e.g.

for the *Nil*, *Solv* and $\widetilde{SL_2\mathbb{R}}$ cases) do not correspond to actual flat spacetimes with spacelike slices.

For comparison with theorem 1.1, we recall the corresponding statement from the de Sitter case (particular examples illustrating this case have appeared in the physics literature; see [2, 7, 22]). A manifold is *conformally flat* if it admits a (locally) conformally flat Riemannian metric; for 3-manifolds this is equivalent [25] to the existence of a *flat conformal* or *Möbius* structure.

Theorem 1.4 (See [25]). *M is a spacelike slice of a de Sitter spacetime if and only if M is conformally flat.*

Together these results show that there are many more possibilities for slices of de Sitter spacetimes than for flat spacetimes: for example, all manifolds modelled on the constant-curvature or product model spaces are conformally flat. Also notable is the fact that the connected sum of conformally flat manifolds is conformally flat [19] and even some manifolds modelled on $\widetilde{SL_2\mathbb{R}}$ are conformally flat [13], providing large classes of examples not present in the flat case. Unfortunately, no classification of conformally flat 3-manifolds is known in general, even assuming the geometrization conjecture. The best results in this direction are due to Kapovich (see, for instance, [17]).

It should be emphasized that the statement of the main theorem is purely topological—it says nothing about how a given M arises as a spacelike slice, its induced Riemannian metric, or about the moduli space of spacetimes having a given M as a Cauchy surface. In the de Sitter case, these questions were worked out completely in [26]: the moduli space of de Sitter domains of dependence $M \times \mathbb{R}$ is identified with an appropriate deformation space of conformally flat metrics on M, thus reducing the question to a widely studied problem in Riemannian geometry. The classification theorems in [20, 26] basically come from a convexity result for the causal horizon of a spacelike slice, followed by a careful analysis of the geometric structure of the causal horizon. Our goal in writing this paper, in contrast, was to derive as much as possible purely from results in 3-manifold topology, hopefully returning to a study of the causal horizon in a subsequent paper.

An added bit of information which falls out in the course of the proof of theorem 1.1 is the determination of all possible holonomy homomorphisms $\pi_1(M) \to ISO(3, 1)$. Of course this falls well short of describing the moduli space of flat spacetimes, as it is not *a priori* true that constant-curvature spacetimes $M \times \mathbb{R}$ are parametrized by their holonomy homomorphisms (indeed, this is false in the de Sitter case [25, 26] as there are infinite families of solutions with identical holonomy representations; Witten remarks on the possible physical significance of this in [34, section 6]). Furthermore, one needs to beware of certain holonomy homomorphisms which are inadmissible because they only arise from spacetimes homeomorphic to $M \times \mathbb{R}$ where the slices $M \times \{t\}$ are not spacelike.

2. Basic results

Let X be a Riemannian or Lorentzian homogeneous space, and let G be its isometry group. If V is a smooth manifold of the same dimension as X, then we can define a (G, X)-*structure* on V to be a maximal atlas of coordinate charts $\{\phi_\alpha : U_\alpha \to X\}$ on V such that the transition functions are given by the action of elements of G. From this data, a standard argument constructs a (G, X)-structure on the universal cover $\widetilde{V}$, a *developing map* $\mathcal{D} : \widetilde{V} \to X$, and a *holonomy homomorphism* $\phi : \pi_1(V) \to G$ satisfying the following equivariance condition:

$$\mathcal{D}(\gamma \cdot x) = \phi(\gamma) \cdot \mathcal{D}(x)$$

for all $\gamma \in \pi_1(V)$ and $x \in \widetilde{V}$ (see [12] for a nice discussion of these notions). One can show that the existence of a (G, X)-structure is equivalent to the existence of a Riemannian or Lorentzian metric everywhere locally isometric to the model space X; in particular, one could rephrase our discussion of the geometrization conjecture in this language.

Let $M \hookrightarrow N$ be a spacelike slice of a flat spacetime. Since M has a trivial normal bundle, we might as well assume that N is homeomorphic to $M \times (0, 1)$. The ideas just introduced provide us with a developing map $\mathcal{D} : \widetilde{N} \to \mathbb{R}^4_1$ and a holonomy homomorphism $\phi : \pi_1(N) \to ISO(3, 1)$ satisfying the equivariance condition above. Here $ISO(3, 1)$ denotes the full isometry group of $\mathbb{R}^4_1$ which we will describe in detail momentarily. We compose these functions with the inclusions $\widetilde{M} \hookrightarrow \widetilde{N}$ and $i_* : \pi_1(M) \overset{\cong}{\to} \pi_1(N)$, respectively, to obtain a spacelike immersion $\mathcal{D} : \widetilde{M} \to \mathbb{R}^4_1$ and a *holonomy group* $\Gamma = \phi(i_*\pi_1(M)) \subset ISO(3, 1)$.

Our first result is that this immersion is actually an embedding and that $\widetilde{M} \cong \mathbb{R}^3$. The proof given here is specific to Minkowski space (the analogous result is false for de Sitter space). Similar theorems are obtained for general classes of spacetimes by Harris in [14, 15].

Lemma 2.1. *The image of* $\mathcal{D} : \widetilde{M} \to \mathbb{R}^4_1$ *is a graph over* $\mathbb{E}^3 = \{\boldsymbol{v} \in \mathbb{R}^4_1 \mid v_4 = 0\}$. *Any two points of* $\mathcal{D}(\widetilde{M})$ *are spacelike separated.*

Proof. First observe that the induced Riemannian metric on the spacelike slice $\widetilde{M}$ is complete since it is the lift of a metric on a closed manifold. The composition of $\mathcal{D}$ and the obvious projection $p : \mathbb{R}^4_1 \to \mathbb{E}^3$ is locally distance-increasing. Therefore, the pullback via $p \circ \mathcal{D}$ of the Euclidean metric from $\mathbb{E}^3$ to $\widetilde{M}$ is pointwise larger than a complete Riemannian metric, so it too is complete. The map $p \circ \mathcal{D}$ is an isometric immersion with respect to this pulled-back metric, so a standard result in Riemannian geometry [18, p 176] implies that $p \circ \mathcal{D}$ is a covering map (and hence a diffeomorphism since $\mathbb{E}^3$ is simply connected). Finally, $\mathcal{D}(\widetilde{M})$ is a graph because $\widetilde{M}$ is connected.

If $\boldsymbol{p}$ and $\boldsymbol{q}$ are two points of the image which are null or timelike separated, consider the path between them given by the intersection of $\mathcal{D}(\widetilde{M})$ and an indefinite two-dimensional plane containing $\boldsymbol{p}$ and $\boldsymbol{q}$. The 'secant line' joining $\boldsymbol{p}$ and $\boldsymbol{q}$ in this plane has slope greater than 1, so the mean value theorem implies that some tangent vector to this path is null or timelike, a contradiction. □

In all that follows we will identify $\widetilde{M}$ with its image in $\mathbb{R}^4_1$.

Every isometry of $\mathbb{R}^4_1$ can be written uniquely as $\boldsymbol{x} \mapsto A\boldsymbol{x} + \boldsymbol{b}$, where the *linear part* A lies in $O(3, 1)$ and $\boldsymbol{b} \in \mathbb{R}^4_1$. If we let $L : ISO(3, 1) \to O(3, 1)$ be the homomorphism projecting to the linear part, we have the following short exact sequence:

$$1 \to \mathbb{R}^4_1 \to ISO(3, 1) \xrightarrow{L} O(3, 1) \to 1.$$

Given a subgroup Γ of $ISO(3, 1)$, define $T(\Gamma) = \ker L\big|_\Gamma$; we call $T(\Gamma)$ the *translational subgroup* of Γ. There is a corresponding short exact sequence for Γ

$$1 \to T(\Gamma) \to \Gamma \xrightarrow{L} L(\Gamma) \to 1$$

which is central to our study of possible holonomy groups.

Lemma 2.2. *Let* $\Gamma \subset ISO(3, 1)$ *be the holonomy group of a spacelike slice. Then:*

(a) Γ *is a discrete, torsion-free subgroup of* $ISO(3, 1)$*;*
(b) $T(\Gamma)$ *consists of spacelike vectors and is isomorphic to* $\mathbb{Z}^k$*, for* $k = 0, 1, 2$ *or 3;*
(c) $L(\Gamma)$ *leaves invariant the spacelike subspace spanned by* $T(\Gamma)$.

Proof. The first part is straightforward by lemma 2.1—since $\widetilde{M} \cong \mathbb{R}^3$, Γ has finite cohomological dimension and is therefore torsion-free. Discreteness follows because Γ acts properly discontinuously on $\widetilde{M}$. Because all pairs of points in $\widetilde{M}$ are spacelike-separated, it is clear that $T(\Gamma)$ consists only of spacelike vectors. It is isomorphic to $\mathbb{Z}^k$ since it is a discrete subgroup of $\mathbb{R}^4_1$ and k cannot be 4 since otherwise $T(\Gamma)$ would have to contain a non-spacelike vector. For the last part, if $\gamma_1 \in \Gamma$ is given by $\boldsymbol{x} \mapsto A\boldsymbol{x} + \boldsymbol{b}$ and $\gamma_2 \in T(\Gamma)$ is translation by $\boldsymbol{t}$, then $\gamma_1\gamma_2\gamma_1^{-1}$ is easily computed to be the element $\boldsymbol{x} \mapsto \boldsymbol{x} + A\boldsymbol{t}$ of $T(\Gamma)$. □

Lemma 2.3. *If Γ is a discrete subgroup of $ISO(3,1)$ with $L(\Gamma)$ indiscrete, then $L(\Gamma)$ is virtually solvable.*

Proof. A theorem of Auslander (see [24, theorem 8.24]) says that $\overline{L(\Gamma)}^0$ is always solvable and is non-trivial since $L(\Gamma)$ is indiscrete. The closed, connected, solvable, non-trivial Lie subgroups of $SO(3,1)^0$ are easy to write down as in [5]; in particular, the set of points F on the sphere at infinity $\partial\mathbb{H}^3$ fixed by $\overline{L(\Gamma)}^0$ must consist of one or two points. The stabilizer in $SO(3,1)^0$ of a point at infinity is isomorphic to the group $Sim^+(\mathbb{R}^2)$ of orientation-preserving similarities of $\mathbb{R}^2$, which is a solvable group. Since $L(\Gamma)$ normalizes $\overline{L(\Gamma)}^0$, it leaves F invariant and therefore has a subgroup of index at most two fixing F pointwise, and therefore conjugate into $Sim^+(\mathbb{R}^2)$. The lemma follows. □

The remaining lemmas are well known results from 3-manifold topology. We record those which will be used repeatedly in section 3.

Lemma 2.4 (See [16]). *Suppose $\widetilde{M} \cong \mathbb{R}^3$ and $\mathbb{Z}^3 \subseteq \pi_1(M)$. Then M is finitely covered by the 3-torus T^3.*

Proof. While a much more general result is proved in [16], we give a simpler proof sufficient for our needs. Consider the cover $\widehat{M}$ of M corresponding to the $\mathbb{Z}^3$ subgroup. Since $\widetilde{M} \cong \mathbb{R}^3$, $\widehat{M}$ is a $K(\mathbb{Z}^3, 1)$ and hence is homotopy equivalent to T^3. This implies that $H_3(\widehat{M}) \cong H_3(T^3) \cong \mathbb{Z}$, thus $\widehat{M}$ is closed and the covering $\widehat{M} \to M$ is finite. Finally, appealing to [31], we have $\widehat{M} \cong T^3$. □

The next result explains the topological structure of manifolds modelled on $\mathbb{E}^3$ and $\mathbb{H}^2 \times \mathbb{R}$. It connects theorem 1.1 and corollary 1.2 and will also be used in section 3.

Lemma 2.5 (See [27]). *M is modelled on $\mathbb{E}^3$ if and only if it is finitely covered by the 3-torus, and M is modelled on $\mathbb{H}^2 \times \mathbb{R}$ if and only if it is finitely covered by $\Sigma \times \mathbb{S}^1$, where Σ is a closed, orientable surface of genus at least two.*

3. Proof of main theorem

Before embarking on the proof of theorem 1.1 we note that its main content is the 'only if' part, which excludes many kinds of 3-manifolds from being spacelike slices of flat spacetimes. In particular, manifolds modelled on five of Thurston's eight geometries cannot be spacelike slices; it is useful in traversing the proof to keep some of these exclusions in mind. For instance, it follows immediately from lemma 2.1 that manifolds modelled on $\mathbb{S}^3$ or $\mathbb{S}^2 \times \mathbb{R}$ do not arise. An important element of the proof of the main theorem is to show that the Euler number of a Seifert fibre space which is a spacelike slice must be zero, excluding manifolds modelled on *Nil* or $\widetilde{SL_2\mathbb{R}}$. *Solv* and *Nil* are interesting since it is possible to find flat spacetimes homeomorphic to $M \times \mathbb{R}$ where M is modelled on *Solv* or *Nil*, but the main theorem says that the slices $M \times \{t\}$ can never be spacelike.

Proof of theorem 1.1. The 'if' half is easy as each of the spaces $\mathbb{H}^3$, $\mathbb{E}^3$ and $\mathbb{H}^2 \times \mathbb{R}$ embeds in $\mathbb{R}^4_1$ (though there can be geometrically distinct embeddings as we will see for $\mathbb{E}^3$). We have

- $\mathbb{H}^3 = \{\boldsymbol{v} \in \mathbb{R}^4_1 | \langle \boldsymbol{v}, \boldsymbol{v} \rangle = -1, v_4 > 0\}$ (see figure 1)
- $\mathbb{E}^3 = \{\boldsymbol{v} \in \mathbb{R}^4_1 | v_4 = 0\}$
- $\mathbb{H}^2 \times \mathbb{R} = \{\boldsymbol{v} \in \mathbb{R}^4_1 | v_2^2 + v_3^2 - v_4^2 = -1, v_4 > 0\}$ (see figure 2).

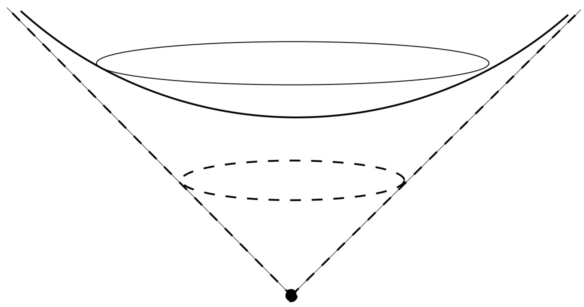

Figure 1. The domain of dependence generated by $\widetilde{M} = \mathbb{H}^3$ is the future of a point.

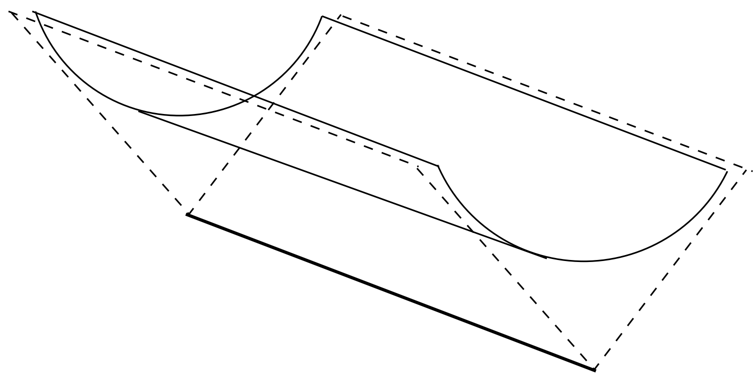

Figure 2. The domain of dependence generated by $\widetilde{M} = \mathbb{H}^2 \times \mathbb{R}$ is the future of a spacelike line.

If M is modelled on a model space X, then it can be realized as $M \cong X/\Gamma$, where Γ is a discrete, cocompact subgroup of the isometry group $Isom(X)$. It is easy to see that if X is one of the three examples above embedded in $\mathbb{R}^4_1$, then there is a corresponding embedding of its isometry group in $ISO(3, 1)$, and that any discrete subgroup of $Isom(X)$ acts discontinuously on a regular neighbourhood of X in $\mathbb{R}^4_1$. The quotient of a small regular neighbourhood is therefore a flat spacetime containing $M \cong X/\Gamma$ as a spacelike slice, as desired. We emphasize in figure 3 that for M hyperbolic, $\widetilde{M}$ need not coincide with the hyperboloid of figure 1.

For the 'only if' half, let M be a spacelike slice of a flat spacetime with holonomy group $\Gamma \subset ISO(3, 1)$. The proof is broken down into four cases, depending on the rank of the translational subgroup $T(\Gamma)$ (lemma 2.2). Note that the cases become easier as we go along, because the presence of a large normal Abelian subgroup of $\pi_1(M)$ typically puts strong topological constraints on M.

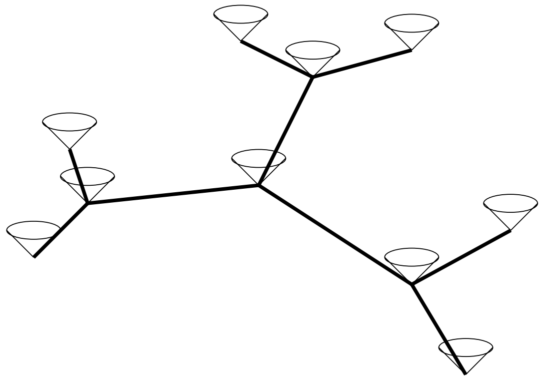

Figure 3. When M is hyperbolic, $\widetilde{M}$ need not coincide with $\mathbb{H}^3$; indeed, the domain of dependence generated by $\widetilde{M}$ is often the future of an infinite-valence spacelike tree. We have not attempted to draw $\widetilde{M}$.

Case 0. Suppose $T(\Gamma) = 0$. This means that L injects Γ into $O(3, 1)$, i.e. $\Gamma \cong L(\Gamma)$. Our assumption that spacelike slices have trivial normal bundles means that $L(\Gamma)$ actually lies in the orthochronous subgroup $O_{\uparrow}(3, 1)$ which coincides with the full isometry group of $\mathbb{H}^3$. Now $L(\Gamma)$ is either discrete or indiscrete. If it is discrete, then it is also cocompact for cohomological reasons. Since M is aspherical, it is homotopy equivalent to the closed hyperbolic 3-manifold $\mathbb{H}^3/L(\Gamma)$ and a result of Gabai–Meyerhoff–Thurston [10] implies that M is itself hyperbolic. In fact, it will turn out that this is the only possibility for the holonomy when M is hyperbolic. Thus in all remaining cases we will be proving that M is modelled on $\mathbb{E}^3$ or $\mathbb{H}^2 \times \mathbb{R}$. In light of lemma 2.5, it suffices to show that M has a finite cover homeomorphic to $\Sigma \times \mathbb{S}^1$, where Σ is a closed, orientable surface, $\Sigma \not\cong \mathbb{S}^2$. We will exploit this fact in all that follows by freely passing to finite covers of M without changing notation. Also note that $L(\Gamma)$ is reducible in all remaining cases, yielding corollary 1.3.

If $L(\Gamma)$ is indiscrete, lemma 2.3 shows that $L(\Gamma) \cong \Gamma$ is virtually solvable. By the main result of [6], we may pass to a finite cover and assume that M is a torus bundle over $\mathbb{S}^1$ with monodromy $\theta : \pi_1(T^2) \to \pi_1(T^2)$ represented by a matrix in $SL(2, \mathbb{Z})$, which by an abuse of notation is also denoted by θ. Clearly, θ has finite order if and only if M is finitely covered by a 3-torus, so we will assume that θ has infinite order. Let $t \in \Gamma$ denote an element inducing the monodromy, i.e. $txt^{-1} = \theta(x)$ for all $x \in \pi_1(T^2)$. The image of the fibre subgroup $\pi_1(T^2) \cong \mathbb{Z} \oplus \mathbb{Z}$ of Γ under L must consist either of elements leaving invariant a geodesic in $\mathbb{H}^3$ (generated by loxodromics or irrational elliptics) or of parabolics with a common fixed point at infinity. In the first case, since t normalizes $\pi_1(T^2)$, $L(t)$ must leave the geodesic invariant (indeed, it must fix it pointwise since $L(t)$ has infinite order). However, this implies that t commutes with $\pi_1(T^2)$, which means that θ is the identity and $\Gamma \cong L(\Gamma) \cong \mathbb{Z}^3$. This contradicts the hypothesis that θ has infinite order, or alternatively shows directly that $M \cong T^3$ in this case by the proof of lemma 2.4.

In the second case, identify $\partial\mathbb{H}^3 \cong \mathbb{C} \cup \{\infty\}$ and conjugate so that the parabolics fix ∞. If θ has a 1-eigenvalue, let g denote an element of $\pi_1(T^2) \cong \mathbb{Z} \oplus \mathbb{Z}$ such that $\theta(g) = g$ and write $L(g)$ as $z \mapsto z + z_0$ for some $z_0 \neq 0 \in \mathbb{C}$. Since t normalizes $\pi_1(T^2)$, $L(t)$ must also fix ∞; write it as $z \mapsto az + b$ for some $a, b \in \mathbb{C}$. However, then with this notation, the relation $[L(t), L(g)] = 1$ becomes $az_0 = z_0$. Thus $a = 1$, $L(t)$ is parabolic, t commutes with $\pi_1(T^2)$, and we are done as above. The final possibility is that θ is of infinite order and has no 1-eigenvalue, in which case it has two real eigenvalues of absolute value not equal to one.

In this case, there is, up to conjugacy, only one possibility for $L(\Gamma)$, and we use this in the appendix to show that Γ cannot be discrete, contradicting lemma 2.2(a).

This completes case 0.

Case 1. Suppose $T(\Gamma) \cong \mathbb{Z}$. We may assume that M is orientable by passing to a double cover. Though we can get away with less, we might as well use the Seifert fibre space theorem of Mess [21], Gabai [9] and Casson and Jungreis [4] which states that a closed, orientable, irreducible 3-manifold whose fundamental group contains a normal $\mathbb{Z}$ is a Seifert fibre space. The proof of this result amounts to showing that the quotient group $\pi_1(M)/\mathbb{Z}$ (namely $L(\Gamma)$ in our set-up) is the fundamental group of a closed 2-orbifold B, which by passing to a finite cover, we can assume to be a closed, orientable surface of genus $g \geqslant 1$. It follows that M is modelled on $\mathbb{E}^3$, $\mathbb{H}^2 \times \mathbb{R}$, *Nil* or $\widetilde{SL_2\mathbb{R}}$. Excluding the final two possibilities amounts to showing that the Euler number e of M is zero.

Our original proof that $e = 0$ amounted to finding the place where the Euler number appears in the Hochschild–Serre sequence for $H^1(\Gamma, \mathbb{R}^4_1)$; we have chosen to give a more geometric argument here. The group Γ has a presentation of the form [23, p 91]

$$\Gamma = \langle a_1, b_1, \ldots, a_g, b_g, h | [a_i, h] = [b_i, h] = 1, h^e = [a_1, b_1] \cdots [a_g, b_g] \rangle;$$

the notation is meant to be obvious: h generates the normal subgroup $T(\Gamma) \cong \mathbb{Z}$ and the other generators project to $L(\Gamma)$, the fundamental group of the base B. Write $\boldsymbol{x} \mapsto \boldsymbol{x} + \boldsymbol{v}$ for the translation h, and $\boldsymbol{x} \mapsto L(a_i)\boldsymbol{x} + t(a_i)$, $\boldsymbol{x} \mapsto L(b_i)\boldsymbol{x} + t(b_i)$ for the other generators. If we let $W = (\mathbb{R}^4_1)^{L(\Gamma)}$ be the subspace left invariant by $L(\Gamma)$, then the commutation relations imply that $L(a_i)\boldsymbol{v} = \boldsymbol{v}$ and $L(b_i)\boldsymbol{v} = \boldsymbol{v}$, and so $\boldsymbol{v} \in W$. Now focus on the final relation: the left-hand side is the translation $\boldsymbol{x} \mapsto \boldsymbol{x} + e\boldsymbol{v}$, while the right-hand side's translational part is a combination of $t(a_i)$ and $t(b_i)$:

$$e\boldsymbol{v} = \sum_{i=1}^{g} L(c_{i-1})(I - L(a_i b_i a_i^{-1}))t(a_i) + L(c_{i-1}a_i)(I - L(b_i a_i^{-1} b_i^{-1}))t(b_i)$$

where $c_j = [a_1, b_1] \cdots [a_j, b_j]$. The linear map being applied to $t(a_i)$ (respectively, $t(b_i)$) in the expression above is called the *Fox derivative* $\frac{\partial R}{\partial a_i}$ (respectively, $\frac{\partial R}{\partial b_i}$) of the usual surface group relator R (see [8, 11]). These partial derivatives can be combined neatly into a single *Fox differential* $\mathrm{d}R : (\mathbb{R}^4_1)^{2g} \to \mathbb{R}^4_1$, in which case

$$e\boldsymbol{v} = \mathrm{d}R(t(a_1), t(b_1), \ldots, t(a_g), t(b_g)).$$

In [11, section 3.7], Goldman has a nice argument showing that the image of $\mathrm{d}R$ is exactly $W^\perp$. Thus $\boldsymbol{v} \in W$ and $e\boldsymbol{v} \in W^\perp$, so

$$0 = \langle \boldsymbol{v}, e\boldsymbol{v} \rangle = e\langle \boldsymbol{v}, \boldsymbol{v} \rangle.$$

Since $\boldsymbol{v}$ is spacelike (this is essential, see [1]), $\langle \boldsymbol{v}, \boldsymbol{v} \rangle \neq 0$, and so $e = 0$.

This completes case 1.

Case 2. Suppose $T(\Gamma) \cong \mathbb{Z}^2$. A theorem in Hempel [16, theorem 11.1] shows that $L(\Gamma)$ has two ends and therefore has a finite index subgroup isomorphic to $\mathbb{Z}$. As usual, we will pass to a finite cover without changing notation and assume $L(\Gamma) \cong \mathbb{Z}$. Stallings' theorem [28] implies that M fibres over the circle with torus fibres. Let $A \in O(3, 1)$ be a generator of $L(\Gamma)$. It leaves invariant the spacelike $\mathbb{E}^2$ spanned by $T(\Gamma) \subset \mathbb{R}^4_1$, and in fact, acts by linear isometries. Since A normalizes the lattice $T(\Gamma)$, the action of A on $\mathbb{E}^2$ must have finite order. It follows that M is finitely covered by the 3-torus, and we conclude that M is modelled on $\mathbb{E}^3$.

This completes case 2.

Case 3. Suppose $T(\Gamma) \cong \mathbb{Z}^3$. Lemma 2.4 implies that M is finitely covered by the 3-torus and is therefore modelled on $\mathbb{E}^3$.

This completes the proof of the main theorem. $\square$

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Appendix

This appendix contains the proof of a result used in case 0 of the proof of the main theorem. It is relegated to an appendix because it relies on a technical cohomology calculation which would have interrupted the flow of the exposition to an unacceptable degree.

The goal is to dispose of a particular class of solvable groups Γ which are fundamental groups of torus bundles over $\mathbb{S}^1$ with 'hyperbolic' monodromy. We will retain all of the notation used in the proof of the main theorem. In particular, the monodromy of the fibration $\theta : \pi_1(T^2) \to \pi_1(T^2)$ is represented by a matrix $A \in SL(2, \mathbb{Z})$ which has two real eigenvalues, say $\lambda > 1$ and $1/\lambda < 1$. Let x_1 and x_2 be the standard generators of $\pi_1(T^2)$ and write the corresponding parabolics $L(x_j)$ as $z \mapsto z + w_j$, $j = 1, 2$. As in section 3, we write $L(t)$ as $z \mapsto az + b$ for some $a, b \in \mathbb{C}$. The relation $txt^{-1} = \theta(x)$ applied to the two generators collates into the following matrix equation:

$$A \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} aw_1 \\ aw_2 \end{pmatrix}.$$

In other words, a must be an eigenvalue of θ (say, $a = \lambda$), and w_j are the components of the corresponding eigenvector (in particular, they are real and irrationally related). For simplicity, we choose $w_1 = 1$.

Lemma A.1. *With $L(\Gamma)$ given as above, Γ must be indiscrete.*

Proof. In fact, we will show that the subgroup $\pi_1(T^2) \subset \Gamma$ must be indiscrete. The idea of the proof is to view Γ as obtained from the indiscrete group $L(\Gamma)$ by adding translations, and to show that there is no way of doing so which makes $\pi_1(T^2)$ discrete. Such a choice of translations is a cocycle in $H^1(\Gamma, \mathbb{R}^4_1)$; that is, a function $c : \Gamma \to \mathbb{R}^4_1$ satisfying the *cocycle relation*

$$c(gh) = c(g) + L(g) \cdot c(h)$$

for all $g, h \in \Gamma$. The coboundaries (change of basepoint) are cocycles of the form

$$c(g) = (1 - L(g))v$$

for some fixed $v \in \mathbb{R}^4_1$. There is, of course, a restriction map to the fibre subgroup

$$H^1(\Gamma, \mathbb{R}^4_1) \to H^1(\pi_1(T^2), \mathbb{R}^4_1)^{\langle t \rangle},$$

where the notation is meant to indicate that restricted classes are invariant under the action of t:

$$[t \cdot c] = [c] \qquad \text{where} \quad (t \cdot c)(x) := L(t)c(\theta^{-1}(x)). \tag{A1}$$

The lemma will follow from the fact that the group $H^1(\pi_1(T^2), \mathbb{R}_1^4)^{\langle t \rangle}$ vanishes. The cocycle relation applied to $x_1x_2 = x_2x_1$ means that for any cocycle $c \in H^1(\pi_1(T^2), \mathbb{R}_1^4)$ we must have

$$(1 - L(x_2))c(x_1) = (1 - L(x_1))c(x_2). \tag{A2}$$

Equation (A1) (really two equations for x_1 and x_2) and equation (A2) are linear in $c(x_1)$ and $c(x_2)$ and can be solved explicitly by choosing a basis for $\mathbb{R}_1^4$ and elements of $O(3, 1)$ representing the generators of $L(\Gamma)$. For instance, following [5], we can choose the first two basis elements to be null vectors fixed by $L(t)$ (the second fixed by $L(x_j)$) and the last two basis elements to be spacelike (the final one also fixed by $L(x_j)$). This yields

$$L(x_1) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad L(x_2) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2}w_2^2 & 1 & w_2 & 0 \\ w_2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$L(t) = \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda^{-1} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Some linear algebra shows that there is a one-dimensional solution set to equations (A1) and (A2); namely by taking $c(x_1) = (0, 1, 0, 0)$ and $c(x_2) = (0, w_2, 0, 0)$. This solution is a coboundary, however, since $c(x_j) = (1 - L(x_j))v$ for $v = (0, 0, -1, 0)$ as a simple calculation shows. Thus $H^1(\pi_1(T^2), \mathbb{R}_1^4)^{\langle t \rangle} = 0$, proving the lemma. □

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