Conformal flows on compact pseudo-Riemannian manifolds

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Motivation

Lichnerowicz conjecture (proved by Lelong-Ferrand): If M is a compact Riemannian manifold and Conf M is not compact, then M is conformally equivalent to the round sphere S^n.

Pseudo-Riemannian Lichnerowicz conjecture: If M is a compact pseudo-Riemannian manifold and Conf M does not preserve any metric in the given conformal class, then M is locally conformally equivalent to flat $\mathbf{R}^{p,q}$.

Einstein spaces $\mathrm{Ein}^{p,q}$

- Q standard quadratic form on $\mathbf{R}^{p+1,q+1}$
- $\mathcal{N}$ nullcone : $\{\mathbf{x} \in \mathbf{R}^{p+1,q+1} \mid Q(\mathbf{x}) = 0\}$
- $\mathrm{Ein}^{p,q} = \mathbf{P}(\mathcal{N})$
- Conf $\mathrm{Ein}^{p,q} = PO(p+1, q+1)$

Rank-1 vs higher-rank dynamics

Source-sink dynamics on S^n:

- any unbounded conformal flow has attracting and repelling fixed points
- near attracting fixed point, some open set is contracted
- same behavior near (approximate) fixed point of unbounded conformal flow in general compact Riemannian manifold

Not so for type (p, q)

On $\mathrm{Ein}^{1,n-1}$, for example, can have attracting and repelling circles.

Simple groups of conformal flows

Note rk $O(p+1, q+1) = p+1$.

Setup:

- M compact pseudo-Riemannian manifold, dim $M \geq 3$
- G simple connected subgroup of Conf M

Zimmer: rk $G \leq p+1$.

Bader & Nevo; Frances & Zeghib: If rk $G = p+1$, then M is conformally equivalent to $\mathrm{Ein}^{p,q}$, up to finite covers when $p \geq 2$, up to finite and cyclic covers when $p = 1$.

Note: If $G <$ Isom M, then rk $G \leq p$ (Zimmer)

Nilpotent groups of conformal flows

Fact: If $G < O(p+1, q+1)$ is connected and nilpotent, then $d(G) \leq 2p+1$.

Setup:

- M compact pseudo-Riemannian manifold, $\dim M \geq 3$
- G nilpotent connected subgroup of Conf M

Our theorem: $d(G) \leq 2p+1$. When $d(G) = 2p+1$, then there exists open $\emptyset \neq U \subset M$ locally conformally equivalent to $\mathbf{R}^{p,q}$.

Note: If $G <$ Isom M, then $d(G) \leq 2p$ (BFM)