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http://www.math.ethz.ch/~iozzi/sandiego.pdf

http://www.math.ethz.ch/~iozzi/sandiego.ps

$\Sigma_{g,n}$ compact orientable surface of genus g with $n \geq 0$ boundary components

$\Gamma_{g,n} = \pi_1(\Sigma_{g,n})$

GOAL: Study of $\mathrm{Hom}(\Gamma_{g,n}, G)$ where G is a semisimple Lie group with finite center.
More precisely:

- Isolate interesting subsets and understand the tools to do this;
- put in relation these subsets with geometric structures on $\Sigma_{g,n}$.

Example $\partial\Sigma_{g,n} = \emptyset \Rightarrow \Sigma_{g,n} =: \Sigma_g$.

(1) Riemann surface=Σ_g + hyperbolic metric

(2) Any complete simply connected surface with a metric of constant negative curvature -1 is isometric to the Poincaré disk $\mathbb{D}^2$

(1)+(2) $\Rightarrow \Sigma_g \simeq \mathbb{D}^2/\Lambda$
$\pi_1(\Sigma_{g,n}) \simeq \Lambda < \text{Isom}(\mathbb{D}^2) = \text{PU}(1,1)$
discrete and acts properly discontinuously on $\mathbb{D}^2$

$\mathcal{T}_g$: = $\{$hyp. structures on $\Sigma_g\}/\sim$
(Teichmüller space)

$$\mathcal{T}_g \leftrightsquigarrow \left\{ \begin{array}{l} \text{hyp. mtr. on } \Sigma_g \\ \text{(top.)} g \geq 2 \end{array} \right\} \leftrightsquigarrow \left\{ \begin{array}{l} \rho : \Gamma_g \to \mathrm{SU}(1,1) \\ \text{homomorphism, s.t} \\ \rho \text{ is injective and} \\ \rho(\Gamma_g) \text{ discrete} \end{array} \right\}$$

($\leftrightsquigarrow$ = some identifications are needed).

Other characterizations

- $\mathcal{T}_g \simeq \mathbb{R}^{6g-6}$;
- $\mathcal{T}_g$ is a level set of a function

$$\mathrm{T} : \mathrm{Hom}\left(\Gamma_g, \mathrm{PU}(1,1)\right) \to \mathbb{R}$$

with the properties:

  - $|\mathrm{T}(\rho)| \leq C_g$, C_g=constant
  - $\mathrm{T}(\rho)$ computable from the definition of Γ_g in terms of generators.

We say that ρ is maximal if $\mathrm{T}(\rho) = C_g$.

Theorem (Goldman, 1980) Maximal representations of Γ_g correspond exactly to hyperbolic metrics.

HIGHER TEICHMÜLLER THEORY

More generally, $\rho : \Gamma_g \to G$, G Hermitian Lie group (the associate symmetric space $\mathcal{X}$ has a non-denegerate$G$-invariant differential 2-form $\omega_{\mathcal{X}}$)

Partial results:

Toledo ($\mathrm{PU}(1, n)$, 1989), Hernàndez ($\mathrm{PU}(2, n)$, 1991)

Theorem (Burger–I.–Wienhard, 2003, 2006)

G a Hermitian Lie group, $\rho : \Gamma_{g,n} \to G$ maximal $\chi(\Sigma_{g,n}) \leq -1$. Then:

The representation ρ is injective and with discrete image.

Much more can be said:

Theorem (Burger–I.–Wienhard, 2003, 2006)
G a Hermitian Lie group $\rho : \Gamma_{g,n} \to G$ maximal $\chi(\Sigma_{g,n}) \leq -1$. Then:

1. $\overline{\rho(\Gamma_{g,n})}^Z =: L$ is reductive;
2. The symmetric space $\mathcal{Y}$ associated to L is Hermitian of tube type and is isometrically embedded in the symmetric space $\mathcal{X}$ associated to G, $\mathcal{Y} \hookrightarrow \mathcal{X}$;
3. $\rho\big(\pi_1(\Sigma_{g,n})\big)$ leaves invariant a maximal tube type subdomain.

Remark

- $\{$Hermitian symm.$\} = \{$tube type$\} + \{$non-tube type$\}$;
- Most Hermitian symmetric spaces of tube type can arise as $\mathcal{Y}$ for a fixed $\mathcal{X}$.

Partial results

- Bradlow–García-Prada–Gothen if $\partial\Sigma_{g,n} = \emptyset$ (PU(p,q), 2003; SO$^*(2n)$, 2005);
- Koziarz–Maubon (PU$(1,n)$, 2003);
- related results by Hitchin Labourie (if $\partial\Sigma_{g,n} = \emptyset$), and Fock–Goncharov.

TOOLS AND DEFINITIONS

Tools: Bounded continuous cohomology for (locally compact topological) groups (Monod, Burger–Monod, Burger–I., 2000, 2001).

(Continuous) cohomology = quotient of spaces of functions

Bounded (continuous) cohomology = quotient of spaces of bounded functions

$$\mathrm{H}^n_{\mathrm{cb}}(G) := \frac{\ker\{d : \mathrm{C_b}(G^{n+1}) \to \mathrm{C_b}(G^{n+2})\}}{\mathrm{im}\,\{d : \mathrm{C_b}(G^{n}) \to \mathrm{C_b}(G^{n+1})\}}$$

Striking difference $\Gamma_{g,n} \simeq \mathbb{F}_{2g+n-1}$ and

$$\mathrm{H}^2(\Gamma_{g,n}) = 0 \qquad \text{but} \qquad \dim \mathrm{H}^2_{\mathrm{b}}(\Gamma_{g,n}) = \infty$$

Definition of $\mathsf{T}(\rho)$.

1.

$$\Omega^2(\mathcal{X}) \xrightarrow[\text{Van Est}]{\simeq} \mathsf{H}^2_{\mathsf{c}}(G) \xrightarrow[\text{B–M}]{\simeq} \mathsf{H}^2_{\mathsf{cb}}(G)$$

$$\omega_{\mathcal{X}} \xrightarrow[\text{integration on geodesic simplices}]{} \kappa^{\mathsf{b}}_{\mathcal{X}}$$

2. $\rho : \Gamma_{g,n} \to G \Rightarrow \rho^* : \mathsf{H}^2_{\mathsf{cb}}(G) \to \mathsf{H}^2_{\mathsf{b}}(\Gamma_{g,n})$ (obvious definition)
3. $\mathsf{H}^2_{\mathsf{b}}(\Gamma_{g,n}) \simeq \mathsf{H}^2_{\mathsf{b}}(\Sigma_{g,n}) \simeq \mathsf{H}^2_{\mathsf{b}}(\Sigma_{g,n}, \partial\Sigma_{g,n})$ (easy statement + amenability of the fundamental group of the boundary components)
4. $\mathsf{H}^2_{\mathsf{b}}(\Sigma_{g,n}, \partial\Sigma_{g,n}) \to \mathsf{H}^2(\Sigma_{g,n}, \partial\Sigma_{g,n})$
5. duality between relative cohomology and relative homology and evaluation against the relative fundamental class.

Definition of $\mathsf{T}(\rho)$

$$
\begin{array}{ccc}
\omega_{\mathcal{X}} \in \Omega^2(\mathcal{X}) & \longrightarrow & \mathsf{H}^2_{\mathsf{cb}}(G) \\
 & & \big\downarrow \rho^* \\
 & & \mathsf{H}^2_{\mathsf{b}}(\Gamma_{g,n}) \\
 & & \big\downarrow \simeq \\
 & & \mathsf{H}^2_{\mathsf{b}}(\Sigma_{g,n}) \\
 & & \big\downarrow \simeq \\
 & & \mathsf{H}^2_{\mathsf{b}}(\Sigma_{g,n}, \partial\Sigma_{g,n}) \\
 & & \big\downarrow \\
 & & \rho^*(\kappa^{\mathsf{b}}_{\mathcal{X}}) \in \mathsf{H}^2(\Sigma_{g,n}, \partial\Sigma_{g,n})
\end{array}
$$

$$\mathsf{T}(\rho) := \left\langle \rho^*(\kappa^{\mathsf{b}}_{\mathcal{X}}), [\Sigma_{g,n}, \partial\Sigma_{g,n}] \right\rangle$$

Interesting properties

1. If $\Sigma = \Sigma_1 \sharp \Sigma_2$ and $\rho_i : \pi_1(\Sigma_i) \to G$ is the restriction of $\rho : \Sigma \to G$, then

$$\mathsf{T}(\rho) = \mathsf{T}(\rho_1) + \mathsf{T}(\rho_2)$$

2. $\mathsf{T}(\rho)$ takes discrete values if $\partial\Sigma_{g,n} =$ and continuous values otherwise.

Symplectic Anosov Structures

V a real vector space with a symplectic form $\langle \cdot\,, \cdot \rangle$
Γ a hyperbolization of Γ_g corresponding to $\Sigma = \Gamma \backslash \mathcal{D}$
$\rho : \Gamma \to \mathsf{Sp}(V)$ a representation
Construct the flat bundle E^ρ with fiber V over the unit tangent bundle $T^1\Sigma$ of the hyperbolic surface Σ

$$E^\rho := \Gamma \backslash (T^1\mathcal{D} \times V) \to T^1\Sigma$$

The geodesic flow on $T^1\Sigma$ lifts to $T^1\mathcal{D}$ and can be extended to E^ρ (trivially to V)
The bundle E^ρ is endowed with a symplectic form and if $J : T^1\Sigma \to \mathsf{End}(E^\rho)$ is a complex structure, we say that J is positive if

$$\langle \cdot\,, J\cdot \rangle : E^\rho \times E^\rho \to \mathbb{R}$$

is positive definite on each fiber.

Theorem [Burger–I.–Labourie–Wienhard] Assume that ρ is maximal. Then there is a splitting into continuous Lagrangian subbundles

$$E^\rho = E^\rho_+ \oplus E^\rho_-$$

such that

- J interchanges E^ρ_- and E^ρ_+
- for all $t \geq 0$,

$$\|g^\rho_t \xi\| \leq e^{-At}\|\xi\| \text{ for all } \xi \in E^\rho_+$$

and

$$\|g^\rho_{-t}\xi\| \leq e^{-At}\|\xi\| \text{ for all } \xi \in E^\rho_- \,.$$

Corollary Assume that ρ is maximal. Then $\rho : \Gamma_g \to \mathrm{Sp}(V)$ is a quasiisometric embedding.

Classical Simple Groups of Hermitian Type

• $\mathsf{SU}(p,q) := \{g \in \mathsf{SL}(p+q,\mathbb{C}) : g^t I_{p,q}\overline{g} = I_{p,q}\}$
(leaves invariant an hermitian form $\langle z,w\rangle = \overline{\langle w,z\rangle}$)

• $\mathsf{Sp}(2n,\mathbb{R}) := \{g \in \mathsf{GL}(2n,\mathbb{R}) : \langle gx,gy\rangle = \langle x,y\rangle\}$
(symplectic form $\langle z,w\rangle = -\langle w,z\rangle$)

• $\mathsf{SO}^*(2n) := \{g \in \mathsf{SO}(2n,\mathbb{C}) : g^t J_n\overline{g} = J_n\}$
$= \{g \in \mathsf{SL}(2n,\mathbb{C}) : g^t J_n\overline{g} = J_n,\ g^t g = I_{2,n}\}$

• $\mathsf{SO}(2,n) := \{g \in \mathsf{SL}(n+2,\mathbb{C}) : g^t I_{2,n} g = I_{2,n}\}$

where $I_{p.q} = \begin{pmatrix} -I_p & p \\ 0 & I_q \end{pmatrix}$ and $J_n = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$.